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Eigenvalues of Toeplitz Matrices Associated with Orthogonal Polynomials

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A connection between the asymptotic distribution of the zeros of orthogonal polynomials and the asymptotic behavior of the eigenvalues of Toeplitz matrices associated with these orthogonal polynomials is given. The result is applied to various families of orthogonal polynomials.

1. INTRODUCTION

Let α be a positive measure on the real line for which all the moments exist. The support of α is defined as

$$\operatorname{supp}(\alpha) = \{x \in \mathbb{R} \mid \forall\, \varepsilon > 0 \colon \alpha(\,]x-\varepsilon, x+\varepsilon[\,) > 0\}.$$

If $\operatorname{supp}(\alpha)$ is an infinite set, then there exists a unique sequence of polynomials $p_n(x;\alpha) = \gamma_n x^n + \cdots$ $(n = 0, 1, 2, \ldots)$ with $\gamma_n > 0$ such that

$$\int_{-\infty}^{+\infty} p_n(x;\alpha)\, p_m(x;\alpha)\, d\alpha(x) = \delta_{m,n}$$

and these polynomials are said to be orthogonal with measure α. Let g be a real valued α-measurable function such that, for every $k \in \mathbb{N}$,

$$\int_{-\infty}^{+\infty} |x^k g(x)|\, d\alpha(x) < \infty;$$

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then the infinite Toeplitz matrix $T(g;\alpha)$ associated with the orthogonal polynomials $p_n(x;\alpha)$ consists of the entries

$$[T(g;\alpha)]_{ij}=\int_{-\infty}^{+\infty} p_i(x;\alpha)\,p_j(x;\alpha)\,g(x)\,d\alpha(x) \qquad (i,j=0,1,2,...).$$

The truncated matrix $T_n(g;\alpha)$ contains the first n rows and columns of $T(g;\alpha)$:

$$T_n(g;\alpha)=[T(g;\alpha)]_{i,j=0,1,...,n-1}. \tag{1.1}$$

It will be convenient to introduce a modified truncated Toeplitz matrix as

$$T_n^*(g;\alpha)=\left[\int_{-\infty}^{+\infty} p_i(x;\alpha)\,p_j(x;\alpha)\,g\left(\frac{x}{c_n}\right)d\alpha(x)\right]_{i,j=0,...,n-1}, \tag{1.2}$$

where c_n $(n=1,2,...)$ is a given sequence of positive numbers. Of course, when $c_n\equiv 1$ then T_n^* and T_n coincide.

Let us denote the zeros of $p_n(x;\alpha)$ in increasing order by $x_{1,n}<x_{2,n}<\cdots<x_{n,n}$ and the eigenvalues of $T_n^*(g;\alpha)$, which are real since T_n^* is symmetric, by $\Lambda_{1,n}\leqslant\Lambda_{2,n}\leqslant\cdots\leqslant\Lambda_{n,n}$. Note that for $g(x)=x$ one has $\Lambda_{k,n}=x_{k,n}/c_n$. Introduce a sequence of discrete measures μ_n $(n=1,2,3,...)$ on $\mathbb{R}$ by

$$\begin{cases}\mu_n\left(\left\{\dfrac{x_{j,n}}{c_n}\right\}\right)=\dfrac{1}{n}\\[2ex] \mu_n(A)=0 \qquad \text{if } A \text{ contains no } x_{j,n}/c_n.\end{cases} \tag{1.3}$$

If there exists a sequence c_n such that the measures μ_n converge weakly to a probability measure μ then we say that the "contracted zeros" $\{x_{j,n}/c_n \mid j=1,2,...,n\}$ are asymptotically distributed according to μ. Weak convergence of a sequence of probability measures μ_n on $\mathbb{R}^k$ to a probability measure μ on $\mathbb{R}^k$ holds when for every bounded and continuous function f on $\mathbb{R}^k$

$$\lim_{n\to\infty}\int_{\mathbb{R}^k} f(x)\,d\mu_n(x)=\int_{\mathbb{R}^k} f(x)\,d\mu(x) \tag{1.4}$$

and if this holds for every bounded continuous function f then it will also hold for every real bounded measurable function f with discontinuities in a set of μ-measure zero [1, Theorem 5.2(iii), p. 31]. If we denote by $C_K(\mathbb{R}^k)$ all continuous functions that vanish outside a compact subset of $\mathbb{R}^k$ then weak convergence also holds if and only if (1.4) is true for every $f\in C_K(\mathbb{R}^k)$ [1, Problem 7, p. 41].

We are interested in the asymptotic distribution of the eigenvalues of $T_n^*(g;\alpha)$. In a way similar to (1.3) we define discrete measures ν_n ($n = 1, 2, , 3, \ldots$) by

$$\begin{cases} \nu_n(\{A_{j,n}\}) = \dfrac{k}{n} & \text{if } A_{j,n} \text{ has multiplicity } k \\ \nu_n(A) = 0 & \text{if } A \text{ contains no } A_{j,n}. \end{cases} \tag{1.5}$$

Our main theorem gives a connection between the weak convergence of μ_n and of ν_n.

THEOREM. *Suppose that there exists a sequence c_n such that the measures μ_n given in* (1.3) *have a weak limit μ and such that $(1/c_n)(\gamma_{n-1}/\gamma_n)$ is $o(\sqrt{n})$ (γ_n is the leading coefficient of $p_n(x;\alpha)$). If g is a bounded measurable function whose points of discontinuity form a set of μ-measure zero, then ν_n as defined in* (1.5) *will converge weakly to μg^{-1}, for which*

$$\mu g^{-1}(A) = \mu(g^{-1}(A))$$

for every Borel set A.

In particular, it follows that whenever the conditions of the theorem are fulfilled,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} G(A_{j,n}) = \int_{-\infty}^{+\infty} G(g(x))\, d\mu(x)$$

for every continuous function G. Grenander and Szegö [5] considered the case $c_n \equiv 1$ and they call the measure μ a canonical distribution. These authors formulate this theorem for the case $\operatorname{supp}(\alpha) = [-1, 1]$ [5, p. 116] but as indicated by Nevai [9, Chap. 5, p. 49] their proof is not correct since they use the Gauss–Jacobi mechanical quadrature with n nodes for polynomials of degree larger than $2n-1$. Nevai [10] used the condition of Erdös and Turán, i.e., $\operatorname{supp}(\alpha) = [-1, 1]$ and $\alpha' > 0$ almost everywhere in $[-1, 1]$, and showed that for the eigenvalues of $T_n(g;\alpha)$, with $g \in L^\infty$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} G(A_{k,n}) = \frac{1}{\pi} \int_{-1}^{1} \frac{G(g(x))}{\sqrt{1-x^2}}\, dx$$

for every continuous function G. For an alternative proof we refer to a paper by Maté, Nevai, and Totik [6]. The main purpose of this paper is to generalize this result to other sets than $[-1, 1]$, including orthogonal polynomials on infinite intervals. In order to do this it was necessary to impose stronger conditions on the function g than $g \in L^{\infty}$: we always suppose that g is a bounded measurable function that is μ-almost everywhere continuous. The main theorem will be proved in the next Section by techniques very similar to those used by Nevai [9, Chap. 5; 10]. In Section 3 we will apply the theorem to some relevant families of orthogonal polynomials.

2. Proof of the Theorem

We will first give some relations between the asymptotic distribution of the contracted zeros $\{x_{j,n}/c_n\}$ and some asymptotic properties of the orthogonal polynomials $p_n(x; \alpha)$. These results will be of use when we return to the proof of our main theorem.

Lemma 1. *Let γ_n be the leading coefficient of $p_n(x; \alpha)$ and c_n be a sequence of positive real numbers such that $(1/c_n)(\gamma_{n-1}/\gamma_n)$ is $o(\sqrt{n})$. Then for every $f \in C_K(\mathbb{R})$*

$$\lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{j=1}^{n} f\left(\frac{x_{j,n}}{c_n}\right) - \frac{1}{n} \int_{-\infty}^{+\infty} f\left(\frac{x}{c_n}\right) \sum_{j=0}^{n-1} p_j^2(x; \alpha)\, d\alpha(x) \right\} = 0. \tag{2.1}$$

Proof. Define another sequence of positive measures ξ_n $(n = 1, 2, ...)$ by

$$\xi_n(A) = \frac{1}{n} \int_A \sum_{j=0}^{n-1} p_j^2(c_n x; \alpha)\, d\alpha(c_n x) \tag{2.2}$$

for every Borel set A. Clearly every ξ_n is a probability measure on $\mathbb{R}$ and also every μ_n, given in (1.3), is a probability measure on $\mathbb{R}$. The signed measure $\mu_n - \xi_n$ is therefore of bounded variation and the total variation of $\mu_n - \xi_n$ is bounded by 2. The relation (2.1) is now equivalent to

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} f(x)\, d(\mu_n - \xi_n)(x) = 0$$

for every $f \in C_K(\mathbb{R})$.

Recall the fundamental polynomials of Lagrange interpolation

$$L_{k,n}(x) = \frac{\gamma_{n-1}}{\gamma_n} \lambda_{k,n}\, p_{n-1}(x_{k,n}; \alpha) \frac{p_n(x; \alpha)}{x - x_{k,n}}. \tag{2.3}$$

The numbers $\{\lambda_{k,n}; j = 1, ..., n\}$ are the Christoffel numbers for the polynomial $p_n(x, \alpha)$ and are defined to be the unique positive numbers for which the Gauss–Jacobi mechanical quadrature

$$\int_{-\infty}^{+\infty} \pi(x)\, d\alpha(x) = \sum_{j=1}^{n} \lambda_{j,n} \pi(x_{j,n}) \tag{2.4}$$

holds for every polynomial π of degree at most $2n-1$ [13, Theorem 3.4.1]. By means of the Christoffel–Darboux formula [13, Theorem 3.2.2]

$$\frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x;\alpha)\, p_{n-1}(y;\alpha) - p_{n-1}(x;\alpha)\, p_n(y;\alpha)}{x-y} = \sum_{j=0}^{n-1} p_j(x;\alpha)\, p_j(y;\alpha) \tag{2.5}$$

one easily obtains

$$L_{k,n}(x) = \lambda_{k,n} \sum_{j=0}^{n-1} p_j(x;\alpha)\, p_j(x_{k,n};\alpha). \tag{2.6}$$

Combining (2.6) and (2.4) gives [3, p. 25]

$$\sum_{k=1}^{n} \frac{1}{\lambda_{k,n}} L_{k,n}^2(x) = \sum_{j=0}^{n-1} p_j^2(x;\alpha)$$

so that

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} f(x)\, d(\mu_n - \xi_n)(x) \right| \\
&= \left| \frac{1}{n} \sum_{j=1}^{n} f\left(\frac{x_{j,n}}{c_n}\right) - \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_{j,n}} \int_{-\infty}^{\infty} f(x)\, L_{j,n}^2(c_n x)\, d\alpha(c_n x) \right| \\
&\leqslant \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_{j,n}} \left\{ \int_{|x-(x_{j,n}/c_n)|<\varepsilon} + \int_{|x-(x_{j,n}/c_n)|\geq\varepsilon} \right\} \\
&\quad \times \left| f\left(\frac{x_{j,n}}{c_n}\right) - f(x) \right| L_{j,n}^2(c_n x)\, d\alpha(c_n x) \\
&= I_1 + I_2.
\end{aligned}$$

Easy estimation gives

$$I_1 = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_{j,n}} \int_{|x-(x_{j,n}/c_n)|<\varepsilon} \left| f\left(\frac{x_{j,n}}{c_n}\right) - f(x) \right| L_{j,n}^2(c_n x)\, d\alpha(c_n x)$$

$$\leqslant \omega_f(\varepsilon) \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_{j,n}} \int_{-\infty}^{\infty} L_{j,n}^2(x)\, d\alpha(x) = \omega_f(\varepsilon),$$

where

$$\omega_f(\varepsilon) = \sup_{|x-y|<\varepsilon} |f(x) - f(y)|$$

is the modulus of continuity of f. Also, by means of (2.3)

$$I_2 = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_{j,n}} \int_{|x-(x_{j,n}/c_n)| \geqslant \varepsilon} \left| f\left(\frac{x_{j,n}}{c_n}\right) - f(x) \right| L_{j,n}^2(c_n x)\, d\alpha(c_n x)$$

$$\leqslant 2\|f\|_\infty \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \frac{1}{n} \sum_{j=1}^{n} \lambda_{j,n} p_{n-1}^2(x_{j,n}; \alpha)$$

$$\times \int_{|x-x_{j,n}| \geqslant \varepsilon c_n} \frac{p_n^2(x;\alpha)}{(x-x_{j,n})^2}\, d\alpha(x)$$

$$\leqslant \frac{2}{n\varepsilon^2} \|f\|_\infty \left(\frac{1}{c_n} \frac{\gamma_{n-1}}{\gamma_n}\right)^2 \sum_{j=1}^{n} \lambda_{j,n} p_{n-1}^2(x_{j,n}; \alpha)$$

$$= \frac{2}{n\varepsilon^2} \|f\|_\infty \left(\frac{1}{c_n} \frac{\gamma_{n-1}}{\gamma_n}\right)^2.$$

Since $(1/c_n)(\gamma_{n-1}/\gamma_n)$ is $o(\sqrt{n})$, it follows that I_2 tends to zero as n tends to infinity. The result follows since ε is arbitrary and $\omega_f(\varepsilon) \to 0$ as ε tends to zero. ▌

LEMMA 2. *Under the same conditions as in the previous lemma one has*

$$\lim_{n\to\infty} \left\{ \frac{1}{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\sum_{j=0}^{n-1} p_j(x;\alpha)\, p_j(y;\alpha) \right)^2 f\left(\frac{x}{c_n}, \frac{y}{c_n}\right) d\alpha(x)\, d\alpha(y) \right.$$

$$\left. - \frac{1}{n} \int_{-\infty}^{+\infty} \sum_{j=0}^{n-1} p_j^2(x;\alpha)\, f\left(\frac{x}{c_n}, \frac{x}{c_n}\right) d\alpha(x) \right\} = 0 \tag{2.7}$$

for every $f \in C_K(\mathbb{R}^2)$.

Proof. Let us use the notation

$$K_n(x, y; \alpha) = \sum_{j=0}^{n-1} p_j(x;\alpha)\, p_j(y;\alpha)$$

and introduce measures β_n ($n = 1, 2, \ldots$) in $\mathbb{R}^2$ by

$$\beta_n(A) = \frac{1}{n} \int_A \int K_n^2(c_n x, c_n y; \alpha)\, d\alpha(c_n x)\, d\alpha(c_n y), \tag{2.8}$$

where A is a Borel set in $\mathbb{R}^2$. Again β_n is a probability measure on $\mathbb{R}^2$ and (2.7) reduces to

$$\lim_{n \to \infty} \left\{ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, d\beta_n(x, y) - \int_{-\infty}^{+\infty} f(x, x)\, d\xi_n(x) \right\} = 0$$

for every $f \in C_K(\mathbb{R}^2)$. We may restrict ourself to functions of the form $f(x)\, g(y)$ with f and g in $C_K(\mathbb{R})$ since linear combinations of such functions are dense in $C_K(\mathbb{R}^2)$ with respect to the uniform topology. Then

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, g(y)\, d\beta_n(x, y) - \int_{-\infty}^{\infty} f(x)\, g(x)\, d\xi_n(x) \right| \\
&= \frac{1}{n} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_n^2(c_n x, c_n y; \alpha) \{ f(x)\, g(y) - f(x)\, g(x) \} \right. \\
&\qquad \left. \times\, d\alpha(c_n x)\, d\alpha(c_n y) \right| \\
&\leqslant \frac{1}{n} \int_{-\infty}^{\infty} \left| f(y) \right| d\alpha(c_n y) \left\{ \int_{|x-y|<\varepsilon} + \int_{|x-y| \geqq \varepsilon} \right\} \\
&\qquad \times |g(x) - g(y)|\, K_n^2(c_n x, c_n y; \alpha)\, d\alpha(c_n x) \\
&= I_1 + I_2.
\end{aligned}$$

Now

$$\begin{aligned}
I_1 &\leqslant \|f\|_\infty\, \omega_g(\varepsilon) \frac{1}{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_n^2(x, y; \alpha)\, d\alpha(x)\, d\alpha(y) \\
&= \|f\|_\infty\, \omega_g(\varepsilon)
\end{aligned}$$

and using the Christoffel–Darboux formula (2.5) leads to

$$\begin{aligned}
I_2 = \frac{1}{n} \left(\frac{1}{c_n} \frac{\gamma_{n-1}}{\gamma_n} \right)^2 &\int_{-\infty}^{\infty} |f(x)|\, d\alpha(c_n x) \int_{|x-y| \geqq \varepsilon} |g(x) - g(y)| \\
&\times \frac{\{ p_{n-1}(c_n x)\, p_n(c_n y) - p_n(c_n x)\, p_{n-1}(c_n y) \}^2}{(x - y)^2}\, d\alpha(c_n y)
\end{aligned}$$

$$\leqslant 2\|f\|_\infty \|g\|_\infty \left(\frac{1}{c_n}\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \frac{1}{n\varepsilon^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{p_{n-1}^2(x)\, p_n^2(y)$$

$$+ p_n^2(x)\, p_{n-1}^2(y) - 2p_{n-1}(x)\, p_n(y)\, p_n(x)\, p_{n-1}(y)\}\, d\alpha(x)\, d\alpha(y)$$

$$= \frac{4}{n\varepsilon^2}\|f\|_\infty \|g\|_\infty \left(\frac{1}{c_n}\frac{\gamma_{n-1}}{\gamma_n}\right)^2.$$

Clearly I_2 tends to zero as n tends to infinity and the result follows since ε is arbitrary and $\omega_g(\varepsilon)\to 0$ as ε tends to zero. ▮

The two previous lemmas are actually results on the weak convergence of the measures ξ_n and β_n given in (2.2) and (2.8), respectively, when the weak convergence of μ_n in (1.3) is known. This can be formulated as follows:

COROLLARY. *If there exists a sequence of positive real numbers c_n such that μ_n, given in* (1.3) *converges weakly to a probability measure μ on $\mathbb{R}$ and such that $(1/c_n)(\gamma_{n-1}/\gamma_n)$ is $o(\sqrt{n})$, then*

$$\text{(a)}\quad \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f\left(\frac{x_{j,n}}{c_n}\right) = \int_{-\infty}^{+\infty} f(x)\, d\mu(x)$$

$$\text{(b)}\quad \lim_{n\to\infty}\frac{1}{n}\int_{-\infty}^{+\infty} f\left(\frac{x}{c_n}\right)\sum_{j=0}^{n-1} p_j^2(x;\alpha)\, d\alpha(x) = \int_{-\infty}^{+\infty} f(x)\, d\mu(x)$$

$$\text{(c)}\quad \lim_{n\to\infty}\frac{1}{n}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f\left(\frac{x}{c_n},\frac{y}{c_n}\right)\left(\sum_{j=0}^{n-1} p_j(x;\alpha)\, p_j(y;\alpha)\right)^2$$

$$\times\, d\alpha(x)\, d\alpha(y) = \int_{-\infty}^{+\infty} f(x,x)\, d\mu(x),$$

where $f(x)$ is a bounded and measurable function on $\mathbb{R}$ with discontinuities on a set of μ-measure zero and $f(x,y)$ is a bounded and measurable function on $\mathbb{R}^2$ for which the discontinuities on the diagonal $\{(x,x)\in\mathbb{R}^2\}$ form a set of μ-measure zero.

Formulas (a) and (b) of this corollary have been proved for $c_n = 1$ by Máté, Nevai and Totik [7] for $\alpha' > 0$ almost everywhere in $[-1, 1]$ and $\operatorname{supp}(\alpha)\cap[-\varepsilon,\varepsilon]$ a finite set for every $\varepsilon > 1$. However, they allowed the more general condition $f\in L^\infty_{d\alpha}$ for Formula (b).

The weak convergence of μ_n does not imply that $(1/c_n)(\gamma_{n-1}/\gamma_n)$ is of the order $o(\sqrt{n})$ so that it is necessary to include this as an extra condition. In Section 3 we shall see that this condition can be checked easily in the most common situations.

The proof of the theorem now proceeds exactly in the same way as Nevai's proof in [10], but uses the above corollary instead of Nevai's Theorem 2.

3. Examples

Let us apply the theorem to some relevant cases. For a definition of various families of orthogonal polynomials we refer to [2, Chap. VI].

(a) Let E be a compact set on the real line. The equilibrium energy of E is defined as

$$V(E) = \inf \left\{ \int_E \int_E \log \frac{1}{|x-y|} \, d\mu(x) \, d\mu(y) | \; \mu \text{ is a probability measure on } E \right\}$$

and the capacity $C(E)$ of E is equal to $\exp(-V(E))$. If the capacity of E is positive, then there exists a unique probability measure μ_E on E such that

$$V(E) = \int_E \int_E \log \frac{1}{|x-y|} \, d\mu_E(x) \, d\mu_E(y)$$

and this measure is called the equilibrium measure (Frostman measure) of E [14, pp. 54–55]. Now suppose that E is a compact set on the real line with positive capacity and such that $\operatorname{supp}(\mu_E) = E$ and let E^* be a bounded and at most denumerable set with accumulation points in E. If α is a positive measure on $E \cup E^*$ such that $\mu_E(\{d\alpha/d\mu_E > 0\}) = 1$, then the sequence of measures μ_n (with $c_n = 1$) converges weakly to the equilibrium measure μ_E [15, 17, 18]. Moreover, since $E \cup E^*$ is compact, there will exist a number B such that $E \cup E^* \subset [-B, B]$ so that

$$\begin{aligned} \left(\frac{\gamma_{n-1}}{\gamma_n} \right)^2 &= \left\{ \int_{-\infty}^{+\infty} x p_{n-1}(x; \alpha) \, p_n(x; \alpha) \, d\alpha(x) \right\}^2 \\ &\leqslant \int_{-B}^{B} x^2 p_n^2(x; \alpha) \, d\alpha(x) \int_{-B}^{B} p_{n-1}^2(x; \alpha) \, d\alpha(x) \\ &\leqslant B^2. \end{aligned}$$

Therefore we find that for the eigenvalues of the truncated Toeplitz matrix $T_n(g; \alpha)$, with g a bounded measurable function that is μ_E-almost everywhere continuous,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} G(\Lambda_{j,n}) = \int_{-\infty}^{+\infty} G(g(x)) \, d\mu_E(x),$$

where G is a continuous function, which means that the eigenvalues $\{\Lambda_{j,n}\}$ are asymptotically distributed according to the measure $\mu_E g^{-1}$. This example contains the Erdös–Turán class of orthogonal polynomials $p_n(x; \alpha)$ with $\operatorname{supp}(\alpha) = [-1, 1]$ and $\alpha' > 0$ almost everywhere in $[-1, 1]$, so that

Jacobi polynomials and Pollaczek polynomials on $[-1, 1]$ are included. Also other sets are allowed as support, such as a finite union of disjoint intervals and various Cantor sets.

(b) Let α be a measure such that supp(α) is a bounded and denumerable set with only one accumulation point at a point $t \in \mathbb{R}$. If we take $c_n \equiv 1$ then it follows from Lemma 2.2 in [17] (with $E_1 = \{t\}$) that μ_n converges weakly to a measure μ which has all its mass at the point t;

$$\begin{aligned} \mu(A) &= 1 \qquad \text{if } t \in A \\ &= 0 \qquad \text{if } t \notin A. \end{aligned}$$

It then follows that for the eigenvalues of the truncated Toeplitz matrix $T_n(g; \alpha)$, with g a bounded measurable function that is continuous at t,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} G(\Lambda_{j,n}) = G(g(t))$$

with G a continuous function. This example contains Tricomi–Carlitz polynomials, modified Lommel polynomials and some q-polynomials.

(c) Consider orthogonal polynomials on the infinite interval $(-\infty, \infty)$. Suppose α is absolutely continuous (with respect to Lebesgue measure) with an even weight function w that is almost everywhere positive and with the following behavior at infinity

$$\lim_{|x| \to \infty} \frac{\log w(x)}{|x|^{\beta}} = -1.$$

It has been shown by Rakhmanov [12] that for $\beta > 1$ one can take

$$c_n = \left(\frac{2n}{\lambda_\beta}\right)^{1/\beta}; \qquad \lambda_\beta = \frac{\Gamma(\beta)}{2^{\beta-2}\{\Gamma(\beta/2)\}^2}$$

and obtain that the measure μ_n converges weakly to a measure μ^β on $[-1, 1]$ for which

$$\mu^\beta(A) = \int_A v(\beta; t)\, dt$$

$$v(\beta; t) = \frac{\beta}{\pi} \int_{|t|}^{1} \frac{y^{\beta-1}}{\sqrt{y^2 - t^2}}\, dy, \qquad -1 \leqslant t \leqslant 1,$$

where A is a Borel set in $[-1, 1]$. The measure μ^β is sometimes called an Ullman measure on $[-1, 1]$. The same result was proven by Mhaskar and

Saff [8] for the weight functions $w(x) = \exp(-|x|^{\beta})$ with $\beta > 0$. The largest zero of the polynomial $p_n(x; \alpha)$ can be estimated by [4]

$$\max_{1 \leqslant k \leqslant n-1} \frac{\gamma_{k-1}}{\gamma_k} \leqslant x_{n,n} \leqslant 2 \max_{1 \leqslant k \leqslant n-1} \frac{\gamma_{k-1}}{\gamma_k}.$$

Rakhmanov [12] has also shown that for $\beta > 1$

$$\lim_{n \to \infty} n^{-1/\beta} x_{n,n} = \left(\frac{2}{\lambda_\beta}\right)^{1/\beta}$$

so that in combination with the previous estimate the boundedness of $(1/c_n)(\gamma_{n-1}/\gamma_n)$ follows. For the eigenvalues of the modified truncated Toeplitz matrix $T_n^*(g; \alpha)$, with g Riemann integrable in $[-1, 1]$, we then obtain

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} G(A_{j,n}) = \int_{-1}^{1} G(g(x))\, v(\beta; x)\, dx$$

with G a continuous function. This example contains Hermite polynomials and in general all weight functions of Freud type, i.e., $w(x) = |x|^{\rho} \exp(-|x|^{\beta})$ $(\rho > -1, \beta > 1)$.

(d) It is well known that orthogonal polynomials $p_n(x; \alpha)$ satisfy a recurrence relation

$$xp_n(x; \alpha) = a_{n+1} p_{n+1}(x; \alpha) + b_n p_n(x; \alpha) + a_n p_{n-1}(x; \alpha) \qquad n = 0, 1, 2, ...$$

with $a_n = \gamma_{n-1}/\gamma_n > 0$ and $b_n \in \mathbb{R}$. If we denote the monic polynomials by $\hat{p}_n(x; \alpha) = (1/\gamma_n)\, p_n(x; \alpha)$, then

$$\hat{p}_{n+1}(x; \alpha) = (x - b_n)\, \hat{p}_n(x; \alpha) - a_n^2 \hat{p}_{n-1}(x; \alpha).$$

If one knows enough about the asymptotic behavior of the recurrence coefficients a_n and b_n (as $n \to \infty$) then it is possible to obtain the asymptotic zero distribution [9, 11, 16]. The order of $(1/c_n)(\gamma_{n-1}/\gamma_n)$ is now easy since this follows immediately from the behavior of a_n. Many families of orthogonal polynomials can be treated by means of the recurrence relation, such as the Poisson–Charlier polynomials, Meixner polynomials and Laguerre polynomials.

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